Notes on Calculating Work using Integrals.

KEY IDEA. Work = (Force)(Displacement) and is an additive quantity.

That is, we can calculate work by calculating bits of work done, adding them all up, and doing $W = \int dW$ with W: work and dW: bit of work done.

There are 2 ways we can slice the work calculation:

(1) We slice/integrate with respect to displacement:

That is, $W = \int F(s)\, ds$ with
s : displacement of object,
ds : small difference in displacement,
$F(s)$: force applied on object at position s

We use this in SPRING PROBLEMS.

(2) We slice/integrate with respect to force:

That is, $W = \int s(F)\, dF$ with
F: force applied on object,
dF: small difference in force,
$s(F)$: displacement of object when force F is applied.

We use this in PUMPING PROBLEMS
and we model dF using the depth x.

Reference for Units of Measurements:

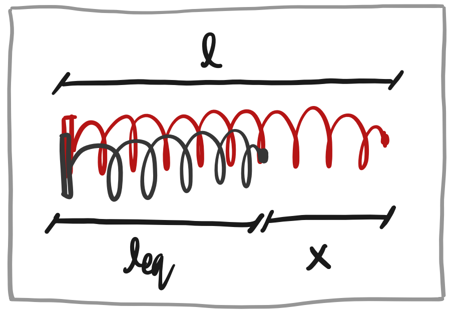
	Quantity	Metric Units	Imperial Units
①	displacement/length	m, meter	ft, feet
②	mass, m	kg, kilogram	lbs, pounds (to avoid confusion with lbf: pound-force, this is sometimes called slug)
③	density, ρ	kg/m^3	lbs/ft^3
④	force, F	$N = \frac{kg \cdot m}{s^2}$, Newton	$lbf = \frac{lbs \cdot ft}{s^2}$, pound-force
⑤	work, W	$J = N \cdot m = \frac{kg \cdot m^2}{s^2}$, Joule	$ft \cdot lbf$, foot-pound

SPRING PROBLEMS.

Let l_{eq}: equilibrium length of spring (at rest),

l: length of spring

x: additional length of spring relative to equilibrium length,
i.e. $x = l - l_{eq}$.



We can set-up the integral with respect to the quantity x.

① By Hooke's law, $F(x) = kx$ with
k: the spring constant
$F(x)$: the force required to keep the spring at length $l_{eq} + x$

② At length $l_{eq} + x$, assume that the work dW done to stretch the spring to length $l_{eq} + x + dx$

is $dW = kx\, dx$ with
kx : force exerted on the spring,
dx : the displacement, i.e. $(l_{eq} + x + dx) - (l_{eq} + x) = dx$

③ We add up all the lengths and get $\boxed{W = \int dW = \int_{x_0}^{x_1} kx\, dx}$ where $x \in [x_0, x_1]$ is the length added to the equilibrium length.

Remarks: (1) Usually, we are given enough information in the problem to find the spring constant k.
Either one of these is given: (-) the value of k is given;

(-) the force exerted by the spring at a certain length is given;
i.e. the force need to hold the spring at that length.
For this, we Hooke's law.

(-) the work done by stretching a spring from one length to another.
For this, we set-up an equation involving an integral.

(2) Hooke's law is usu. stated as $F(x) = -kx$ but that's the force exerted by the spring
i.e. the force opposite of what we apply on the spring.

Example Problem 1

(a) A spring can be stretched 0.2 meters from its equilibrium position with force of 40 N. Determine the spring constant.

Force is given. Use Hooke's law.

$x = 0.2$ m (this is the length added)

$F = 40$ N

By Hooke's law, $F = 40 = kx = 0.2k$; $k = \frac{40}{0.2} = \boxed{200 \frac{N}{m}}$;

(b) The same spring from part (a) has a natural (total) length of 7 meters. The spring is then stretched to the total length of 10 meters. Determine how much work was required to stretch the spring from its natural length of 7 m to a total length of 10 m.

Lengths are given. $l_{eq} = 7$m, $l_0 = 7$ m ; $l_1 = 10$ m

$x_0 = 7 - 7 = 0$ m ; $x_1 = 10 - 7 = 3$ m

$$W = \int_0^3 kx\,dx = \left[\frac{1}{2}kx^2\right]_0^3 = \frac{1}{2}k\left[(3)^2 - (0)^2\right] = \frac{9}{2}k = \frac{9}{2}(200) = \boxed{900 \text{ N}\cdot\text{m}}$$

Example Problem 2

235. A spring has a natural length of 10 cm. It takes 2 J to stretch the spring to 15 cm. How much work would it take to stretch the spring from 15 cm to 20 cm?

Part (a). Find k.

Given: The work done by stretching the spring from 10cm to 15cm is 2J = 2 N·m.

Additional length: $x = 15 - 10 \text{ cm} = 5\text{cm} \cdot \frac{1\text{m}}{100\text{cm}} = 0.05\text{m} = \frac{1}{20}\text{m}$;

Work done : $2 = \int_0^{0.05} kx\,dx = k\left[\frac{1}{2}x^2\right]_0^{0.05} = \frac{1}{2}k\left[\left(\frac{1}{20}\right)^2 - (0)^2\right]$

$2 = \frac{1}{2}k\left(\frac{1}{400}\right)$; $k = 400$ N/m ;

Part (b). Find work going from 15cm to 20cm.

$x_0 = 15 - 10 \text{ cm} = 5\text{cm} = \frac{1}{20}\text{m}$;

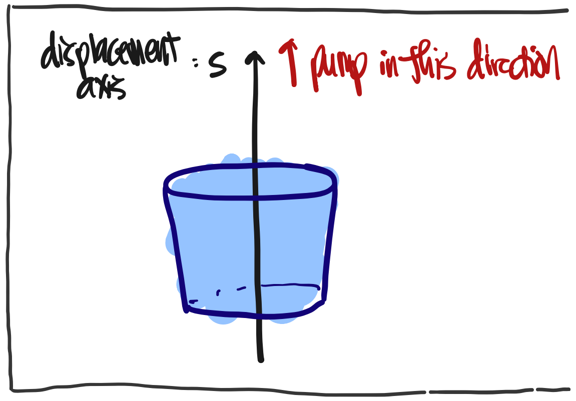
$x_1 = 20 - 10\text{cm} = 10 \text{ cm} \cdot \frac{1\text{m}}{100\text{cm}} = \frac{1}{10}\text{m} = 0.1\text{m}$;

$$W = \int_{0.05}^{0.1} kx\,dx = 400\left[\frac{1}{2}x^2\right]_{0.05}^{0.1} = 200\left[(0.1)^2 - (0.05)^2\right]$$

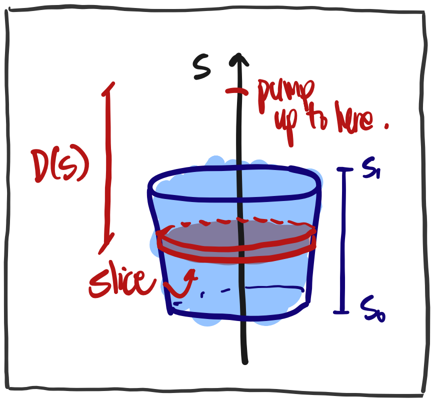
$$= 200(0.0075) = 1.5 \text{ N}\cdot\text{m} = 1.5 \text{ J}$$

PUMPING PROBLEMS

For these problems, we model the volume to be pumped on the displacement axis s, i.e. parallel to the direction of pumping.



We slice perpendicular to the displacement axis and consider the volume dV of the slice.



bounds of volume of liquid : $s \in [s_0, s_1]$

volume of slice : $dV = A(s)\, ds$ with $A(s)$: the cross-sectional area

force needed to lift the slice : $dF = \rho g\, dV = \rho g\, A(s)\, ds$

with ρ: density of slice (ρ is the Greek letter rho)
g: acceleration due to gravity

distance the slice is pumped : some function $D(s)$ with respect to s, usually $D(s) = s + h$ for some constant $h \in \mathbb{R}$

work done by lifting the slice : $dW = \underbrace{D(s)\, dF}_{(\text{displacement})(\text{force})} = D(s)\, \rho g\, dV = \rho g\, D(s) A(s)\, ds$

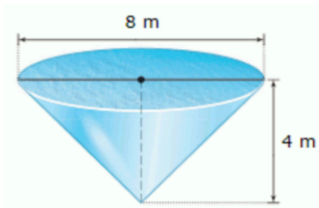
Then, the work W done by pumping liquid is $\boxed{W = \int dW = \rho g \int_{s_0}^{s_1} D(s)\, A(s)\, ds}$

Remarks:
1. The acceleration due to gravity on Earth is $g = 9.81 \frac{m}{s^2} = 32.17 \frac{ft}{s^2}$
2. If the liquid is water, use $\rho_{water} = 1000 \frac{kg}{m^3} = 62.4 \frac{lbs}{ft^3}$
3. Make sure you use the constants that match the given units.

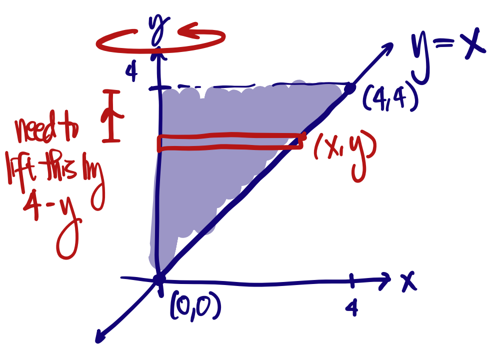
Example Problem.

A full water tank in the shape of an inverted right circular cone is 8 m across the top and 4 m high. How much work is required to pump all the water over the top of the tank? (The density of water is 1000 kg/m^3. Assume g = 9.8 m/s^2.)

J



Part (a). Set-up. Model the slices of the cone as follows:



For $y \in [0,4]$:

Volume of Slice	:	$dV = \pi r^2\, dy = \pi x^2\, dy \overset{x=y}{=} \pi y^2\, dy$
Displacement of the Slice	:	$4-y$
Work needed to lift the Slice	:	$dW = \rho g \left(\text{mass of slice}\right)\left(\text{displacement}\right)$
		$= \rho g (\pi y^2\, dy)(4-y)$

$$W_{pump} = \int_0^4 9800\pi (y^2)(4-y)\, dy \; ;$$

Part (b). Evaluation.

$$W_{pump} = \int_0^4 9800\pi\left(4y^2 - y^3\right) dy = 9800\pi\left[\frac{4}{3}y^3 - \frac{1}{4}y^4\right]_0^4$$

$$= 9800\pi\left[\frac{4}{3}(4)^3 - \frac{1}{4}(4)^4 - (0)\right]$$

$$= 9800\pi\left[\frac{4}{3}(4)^3 - (4^3)\right] = 9800\pi(4^3)\left[\frac{4}{3} - 1\right] = 9800\pi(4^3)\left(\frac{1}{3}\right)$$

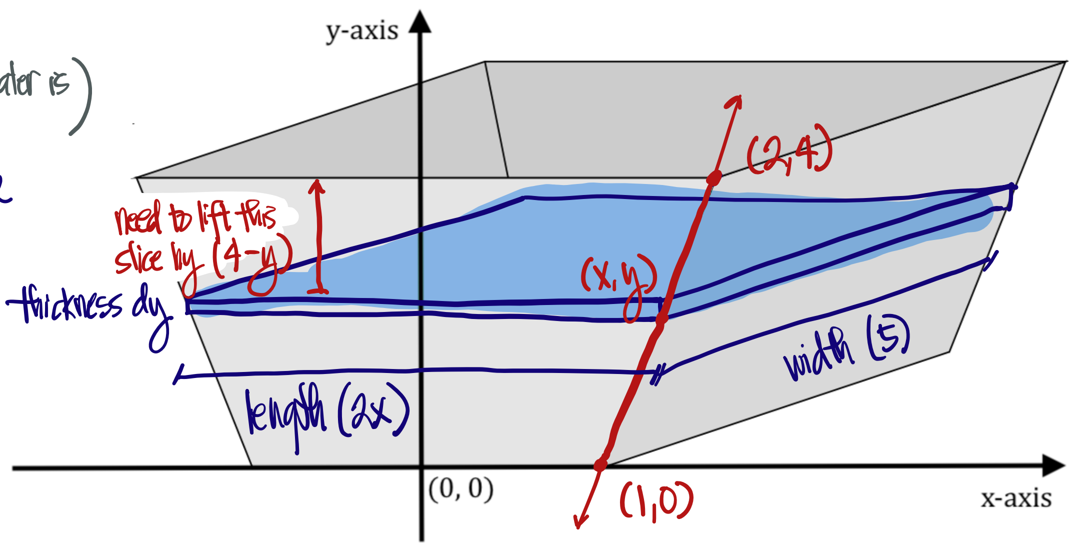
$$= \frac{1}{3}(627\,200\pi)\ \text{N}\cdot\text{m} \approx 656\,802\ \text{J}$$

[Example Problem]

Pumping - Integrating along the y−axis. Consider a full watering trough with an isosceles trapazoidal face. The trough is 4 meters tall, 4 meters wide at the top, 2 meters wide at the bottom, and 5 meters long. Set up AND evaluate the integral needed to determine the work required to pump all the water out over the top edge of the trough. Use the weight density of water $9,800 N/m^3$.

Hint: Imagine the bottom of the face of the trough sitting on the x−axis symmetric about (and parallel to) the y−axis. Just like in problem (4), draw a sample slice, label points, identify the needed formula for the boundary and rewrite it as $x = ...$

We slice this solid (geometry-wise since water is a liquid physically) perpendicular to the lifting axis.



★ We model our slices using the line passing through (1,0) and (2,4).

$$m = \frac{\Delta y}{\Delta x} = \frac{4-0}{2-1} = 4; \quad \text{Line: } y - y_0 = m(x - x_0); \quad y - 0 = 4(x-1); \quad y = 4(x-1)$$

$$\frac{1}{4}y = x - 1; \quad x = \frac{1}{4}y + 1 = \frac{1}{4}(y+4);$$

↑ we solve for x since we have dy as the thickness of our slices.

For $y \in [0,4]$: Volume of the slice : $dV = (2x)(5)\,dy = 10\left(\frac{1}{4}\right)(y+4)\,dy$

Force needed to lift slice : $dF = \rho g\, dV = 9800(10)\left(\frac{1}{4}\right)(y+4)\,dy$

Distance the slice needs to lifted by : $4-y$

Work done by lifting the slice : $dW = 9800(10)\left(\frac{1}{4}\right)(4-y)(4+y)\,dy$

$$W = \int_0^4 9800(10)\left(\frac{1}{4}\right)(4-y)(4+y)\,dy = 98000\left(\frac{1}{4}\right)\int_0^4 16 - y^2\,dy$$

$$= 98000\left(\frac{1}{4}\right)\left[16y - \frac{1}{3}y^3\right]_0^4 = 98000\left(\frac{1}{4}\right)\left[16(4) - \frac{1}{3}(4)^3\right] = 98000\left(\frac{1}{4}\right)\left(\frac{128}{3}\right)$$

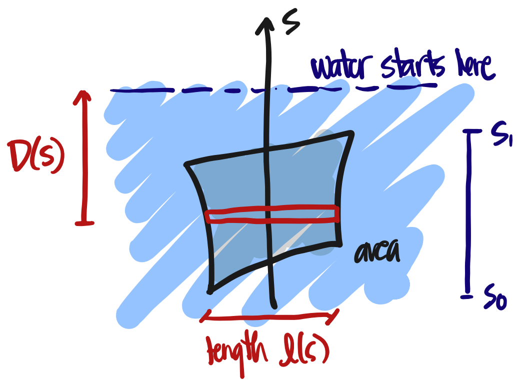
$$= \frac{3\,136\,000}{3}\ \text{N}\cdot\text{m} \approx 1\,045\,333.33\ \text{J} \approx 1045\ \text{kJ}$$

Hydrostatic Force and Pressure.

Key Ideas: The force F applied on an object is $F = (\text{pressure})(\text{area})$. This quantity is additive over area.

For hydrostatic force, we model the problem along the depth axis s (or some axis parallel to the depth).

We slice perpendicular to the depth axis and calculate the force dF applied on the strip dA.



Bounds : $s \in [s_0, s_1]$

Area of Slice : $dA = l(s)\, ds$ with $l(s)$: length of strip

Depth of Slice : Some function $D(s)$,

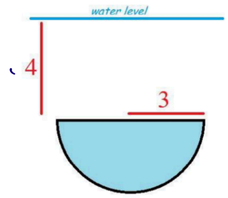
usually $D(s) = s + h$ or $D(s) = -s + h$ depending on the choice of axis.

Hydrostatic Pressure : $P = \rho g D(s)$ with ρ : density of fluid, g : acceleration due to gravity

Hydrostatic Force on Slice : $dF = P\, dA = \rho g D(s)\, l(s)\, ds$

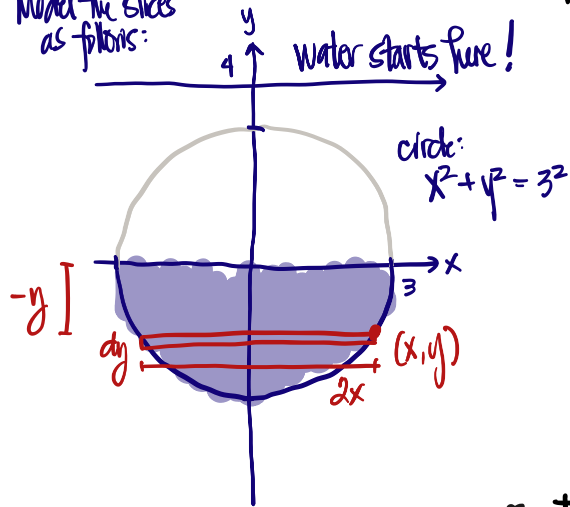
Then, hydrostatic force F on region is $\boxed{F = \int P\, dA = \int_{s_0}^{s_1} \rho g D(s)\, l(s)\, ds}$

Example Problem. **[PHYSICS AND ENGINEERING] Hydrostatic Force.** A vertical semicircular plate is in the wall of a swimming pool. The top of the plate is 4 feet below the water level. The radius of the semicircle is 3 feet. Recall that water has a weight density of 62.5 pounds per cubic foot.



Set up *but DO NOT EVALUATE or even algebraically simplify* an integral that represents the total hydrostatic force on the plate.

Model the slices as follows:



For $y \in [-3,0]$:

Depth of Slice : $4-y$

Hydrostatic Pressure on Slice : $P = \rho g(4-y)$ with $\rho g = 62.5\ \text{lbs}/\text{ft}^3$

Area of Slice : $dA = 2x\,dy = 2\left|-\sqrt{9-y^2}\right|dy = 2\sqrt{9-y^2}\,dy$

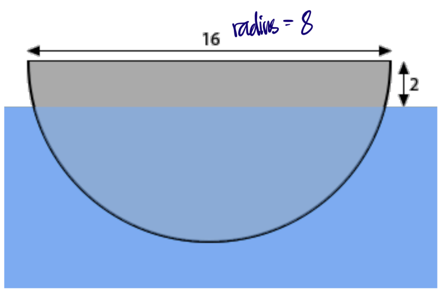
(distance has to be positive)

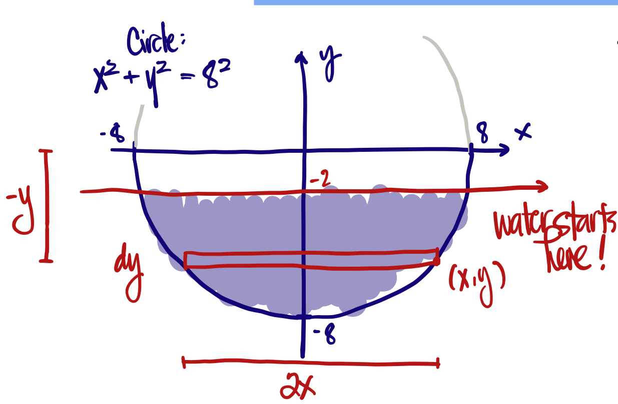
Hydrostatic Force on the Slice : $dF = P\,dA = \rho g(4-y)\left[2\sqrt{9-y^2}\right]dy$

$$F = \int_{-3}^{0} 62.5(4-y)\left[2\sqrt{9-y^2}\right]dy$$

Example Problem. **Hydrostatic Force:** A vertical semi-circular plate is partially submerged in water as shown in the figure below, where the given dimensions are in feet. Recall that water has a weight density of 62.5 pounds per cubic foot.

Set up but *do not evaluate or even algebraically simplify* an integral that represents the total hydrostatic force on the plate.





For $y \in [-8,-2]$:

Depth of the Slice : $(-2)+(-y) = -y-2 = (-1)(y+2)$

Hydrostatic Pressure on the Slice : $P = \rho g(-1)(y+2)$ with $\rho g = 62.5\ \frac{\text{lbs}}{\text{ft}^3}$

Area of the Slice : $dA = 2x\,dy = 2\sqrt{64-y^2}\,dy$

Hydrostatic Force on the Slice : $dF = P\,dA = \rho g(-1)(y+2)2\sqrt{64-y^2}\,dy$

$$F = \int_{-8}^{-2} \rho g(-1)(y+2)2\sqrt{64-y^2}\,dy$$